A Simple Analytical Model for Pre-Congestion Notification (PCN)

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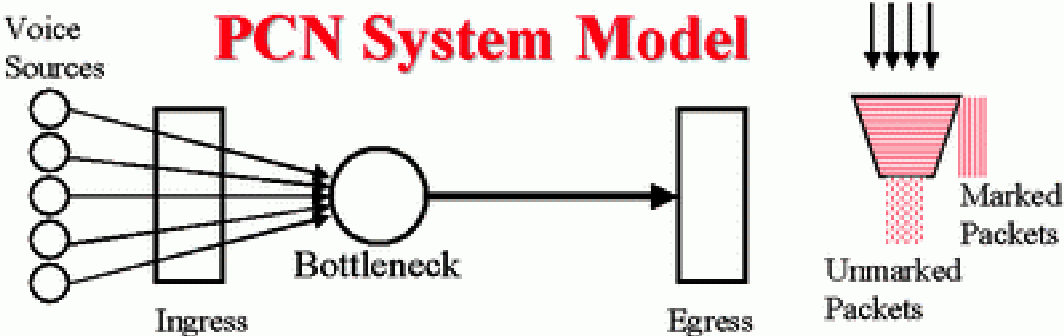
Please check the latest version of these slides at:
http://www.cse.wustl.edu/~jain/ietf/pcn0803.htm

- Advantage of analytical modeling
- Model
- Probability of flow acceptance and flow termination
- Thrashing Index
- Effect of various parameters on thrashing index

Analytical vs. Simulation Models

- Simulation models can be used to model complex scenarios. Analytical models requires simplification.
- Simulation models are limited by the computing capacity. Simulating a few thousand sources may not be practically possible with some packages. Analytical models may or may not have such limitations.
- Studying sensitivity to parameters requires rerunning simulation models many times making the computing problem even worse. Analytical models provide good insight into parameter sensitivity.

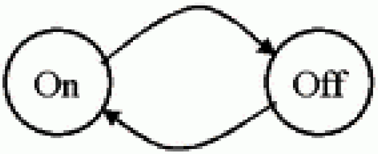
PCN System Model



- *n* voice flows going to through a bottleneck node
- Bottleneck node marks the packets using a token bucket
- Egress node counts the marked packets and communicates the percentage of marked packets to the ingress node
- Ingress node rejects new flows if the percentage of marked packets is above a "rejection threshold"
- Ingress node terminates existing flows if the percentage of marked packets is above a "termination threshold"

Assumptions

1. On-off times of the sources are i.i.d. with exponential distribution



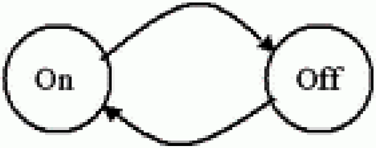
   ⇒ Sources can be modeled as a 2-state Markov Chain
2. The rate of source is constant when it is on
3. Unlimited buffering in the bottleneck ⇒ No Loss
4. The feedback is instantaneous. Propagation delays are not modeled.
5. Single link case – single ingress, single egress.
6. Single marking case

System Parameters and Variables

- n = Number of flows through the bottleneck
- k = Number of flows that are on ⇒ n-k flows are off
- $1/\alpha$ = Average source on-time
- $1/\beta$ = Average source off-time
- p = Fraction of time the source is on = $\alpha/(\alpha+\beta)$
- F = Flow rate in bps when the flow is on
- L = Token bucket rate in bps
- q = probability of a packet being marked
- R = Rejection threshold ⇒ New flows are rejected if $q \geq R$
- T = Termination threshold ⇒ Existing flows are terminated if $q \geq T$
- $x_i(t)$ = Rate of ith source at time t (=F if on, 0 if off)

Notation: Uppercase letters denote fixed parameters.
Lowercase letters denote variables.

2-State Markov Chain Source Model



- i^{th} Flow's rate:

$$x_i = \begin{cases} F & \text{If on} \\ 0 & \text{if off} \end{cases}$$

- Probability of i^{th} flow being on = $p = \frac{\alpha}{\alpha+\beta}$
- Probability k of n flows being on = $\binom{n}{k} p^k (1-p)^{n-k}$

Binomial and Normal Distributions

- For *np*>5, binomial distribution becomes normal with mean *np* and standard deviation $\sqrt{np(1-p)}$

$$\sum_{k=x}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

Binomial

$$\approx 1 - \Phi\left(\frac{x-np}{\sqrt{np(1-p)}}\right)$$

$\Phi(x)$ = Normal CDF

$$\approx \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{x-np}{\sqrt{2np(1-p)}}\right)$$

erf = Error Function

Probability of Rejection

- If there are k active flows: Total load is kF
  - kF-L packets are marked, L packets are not marked
  - % of marked packets wrt L: q = (kF-L)/L = kF/L -1
  - Rejection event happens when % marked ≥ R
  - kF/L-1 ≥ R or k ≥ L(1+R)/For k ≥ k_R

$$k_R = \frac{L(1+R)}{F}$$

- Probability of Rejection Event

$$\sum_{k=k_R}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

$$\approx \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{k_R - np}{\sqrt{2np(1-p)}}\right)$$

- This is also the probability of a new flow being rejected if there are total *n* flows

Example Scenario 1

- On Period = Off period ⇒ p = 0.5
- Per flow rate F = 1
- *Token bucket rate L*= 500
  ⇒ Support 500 active flows (flows that are on)
  ⇒ Support total 1000 flows (includes both on and off flows)
- Rejection Threshold = 0.01
- Termination Threshold = 0.05

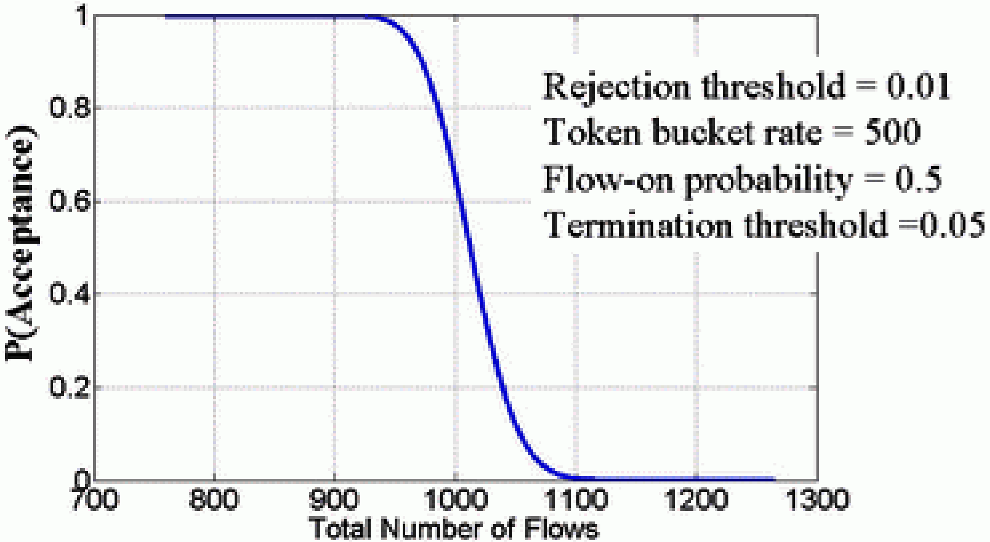
$$k_R = \frac{L(1+R)}{F} = \frac{500(1+0.01)}{1} = 505$$

- Rejection Probability when n = 1050,

$$= \sum_{k=505}^{1050} \binom{1050}{k} 0.5^k(1-0.5)^{1050-k} = 0.9$$

- Flow Acceptance probability = 1 – Flow rejection probability = 0.1

Flow Acceptance Probability



Observation: There is a significant flow acceptance probability even when the number of flows is 10% over the threshold.

Probability of Termination

- If there are k active flows: Total load is kF
  - kF-L packets are marked, L packets are not marked
  - % of marked packets = (kF-L)/L = kF/L-1
  - Rejection event happens when % marked $\geq$ T
  - kF/L-1 $\geq$ T or k $\geq$ L(1+T)/F or k $\geq$ k_T

$$k_T = \frac{L(1+T)}{F}$$

- Probability of termination Event

$$\sum_{k=k_T}^{n} \binom{n}{k} p^k (1-p)^{n-k} \approx \frac{1}{2} - \frac{1}{2}\text{erf}\left(\frac{k_T - np}{\sqrt{2np(1-p)}}\right)$$

- At this event, **multiple flows** may be terminated.

Probability of Termination (Cont)

- The fastest way to bring the system to desired operating range is to terminate $(k-k_T)/p$ flows
  $\Rightarrow$ P(any particular flow being terminated)=$(k-k_T)/np$
- Mean Probability of terminating a particular flow:

$$P = \sum_{k=k_T}^{n} \frac{(k-k_T)}{np} \binom{n}{k} p^k (1-p)^{n-k}$$

- By Gaussian approximation this probability is:

$$P = \int_{k_T}^{\infty} \frac{x-k_T}{\mu} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx.$$

Where $\mu = np, \sigma^2 = np(1-p)$

Flow Termination Probability (Cont)

$$
\begin{aligned}
P &= \frac{1}{\sqrt{2\pi\sigma^2}} \int_{k_T}^{\infty} \frac{x-\mu}{\mu} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} dx \\
&\quad - \frac{k_T-\mu}{\mu}\left[\frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{k_T-\mu}{\sqrt{2\sigma^2}}\right)\right] \\
&= \frac{\sigma^2}{\mu\sqrt{2\pi\sigma^2}} \int_{k_T}^{\infty} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\} d\frac{(x-\mu)^2}{2\sigma^2} \\
&\quad - \frac{k_T-\mu}{\mu}\left[\frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{k_T-\mu}{\sqrt{2\sigma^2}}\right)\right] \\
&= \frac{\sigma}{\mu\sqrt{2\pi}} \exp\left\{-\frac{(k_T-\mu)^2}{2\sigma^2}\right\} \\
&\quad - \frac{k_T-\mu}{\mu}\left[\frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{k_T-\mu}{\sqrt{2\sigma^2}}\right)\right]
\end{aligned}
$$

Example Scenario 1 (Cont)

- P=0.5, L=500, F=1, R=0.01 T=0.05

$$k_T = \frac{L(1+T)}{F} = \frac{500(1+0.05)}{1} = 525$$

- Termination Event Probability when n = 1050,

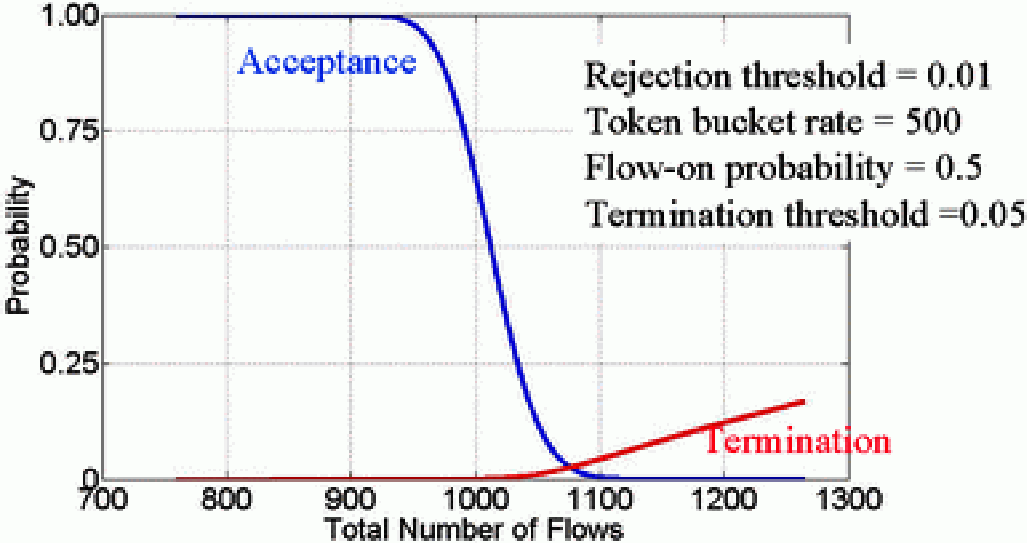
$$= \sum_{k=525}^{1050} \binom{1050}{k} 0.5^k (1-0.5)^{1050-k}$$

$$= 0.5$$

- Flow termination probability when n = 1050

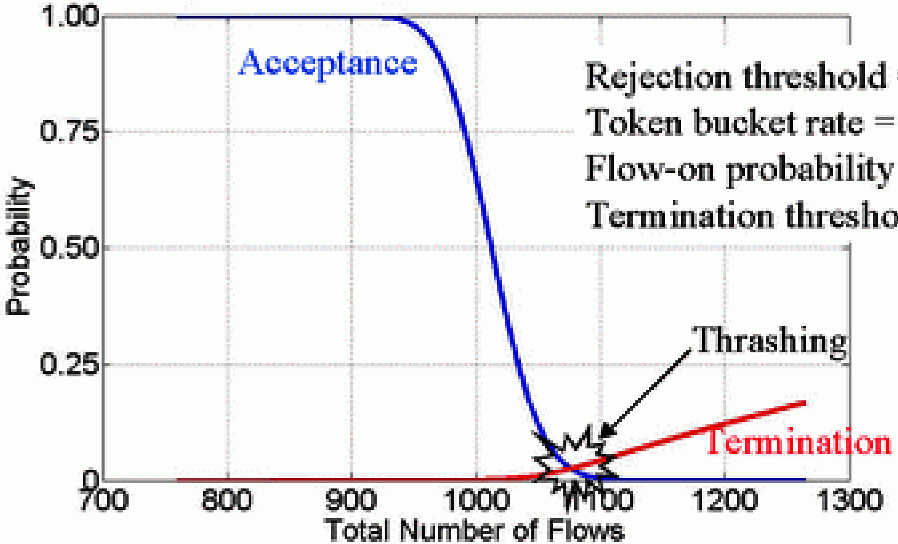
$$= \sum_{k=525}^{1050} \frac{k-525}{0.5} \binom{1050}{k} 0.5^k (1-0.5)^{1050-k}$$

Probability of Terminating Flows



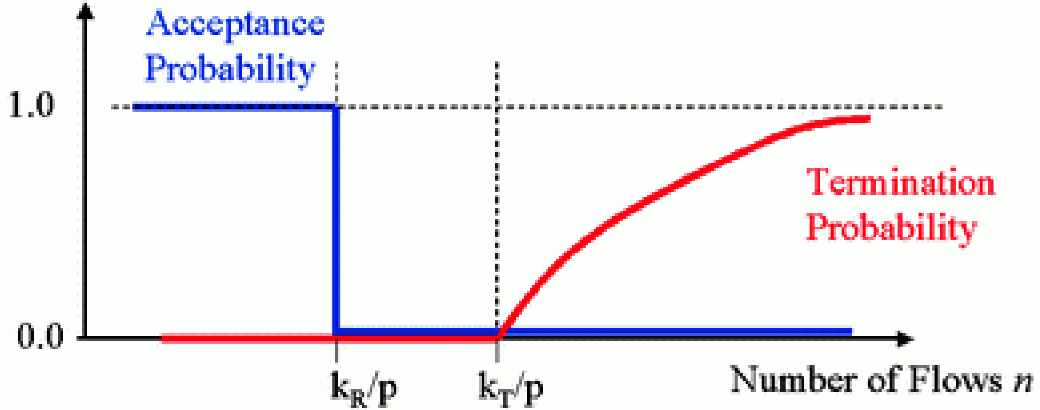
- **Observation**: With 1070 flows, there is 5% probability of accepting a new flow and 5% probability of terminating an existing flow

Probability of Terminating Flows (Cont)



- **Observation**: With 1070 flows, there is 5% probability of accepting a new flow and 5% probability of terminating an existing flow
  $\Rightarrow$ **Thrashing**

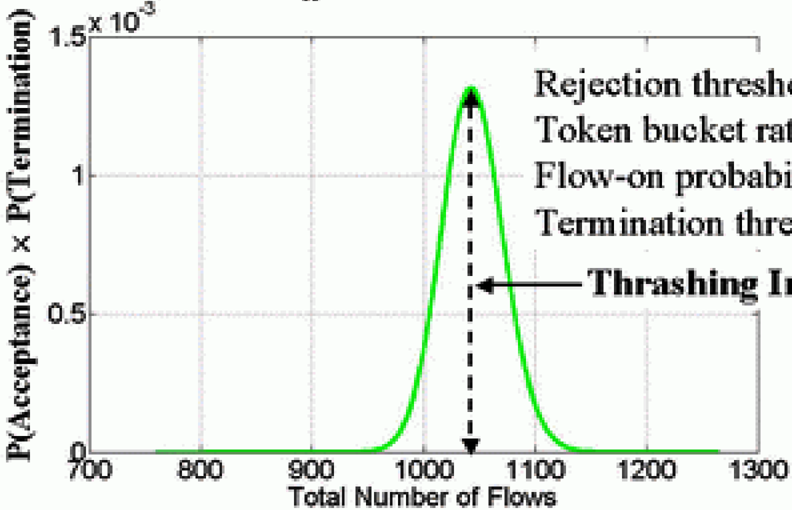
Ideal Desired Behavior



- Every flow should be accepted before the rejection threshold and should be rejected after it.
- No flow should be terminated before the termination threshold and every extra flow should be terminated after the termination threshold

Thrashing Index

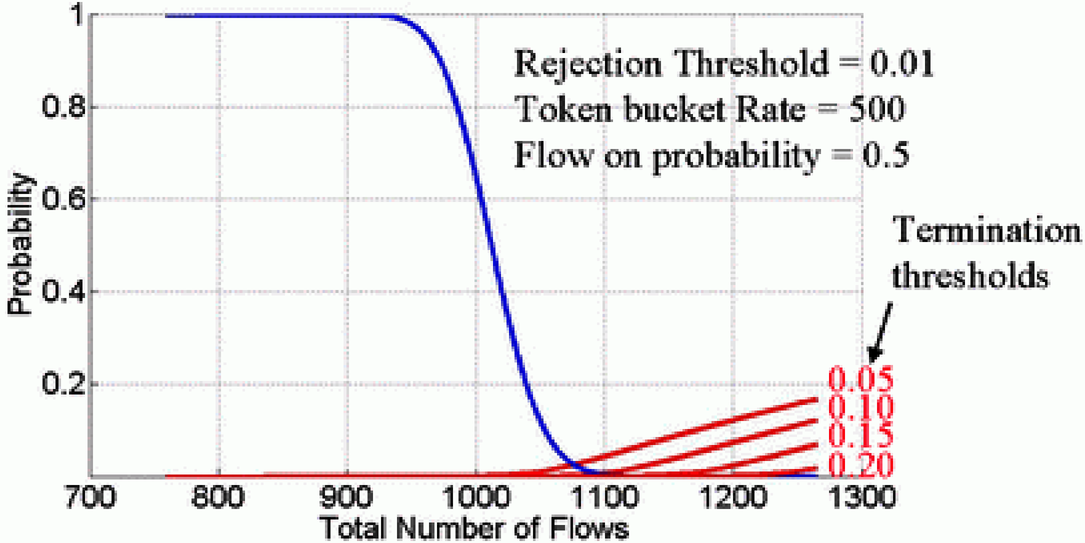
- Ideal: P(Acceptance) × P(Termination) = 0 ∀ n
- Thrashing happens this product is non-zero.
- Thrashing Index = $\max_n$ {P(Acceptance) × P(Termination)}



Sensitivity Analysis

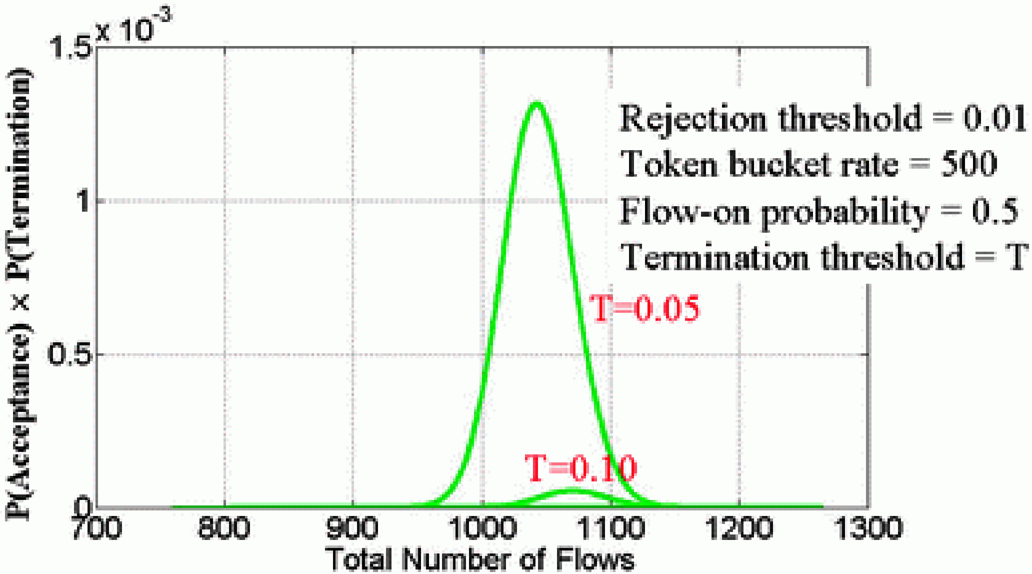
- Sensitivity to:
  - Termination threshold T
  - Flow-On probability p
  - Token bucket Rate L

Sensitivity to Termination Threshold



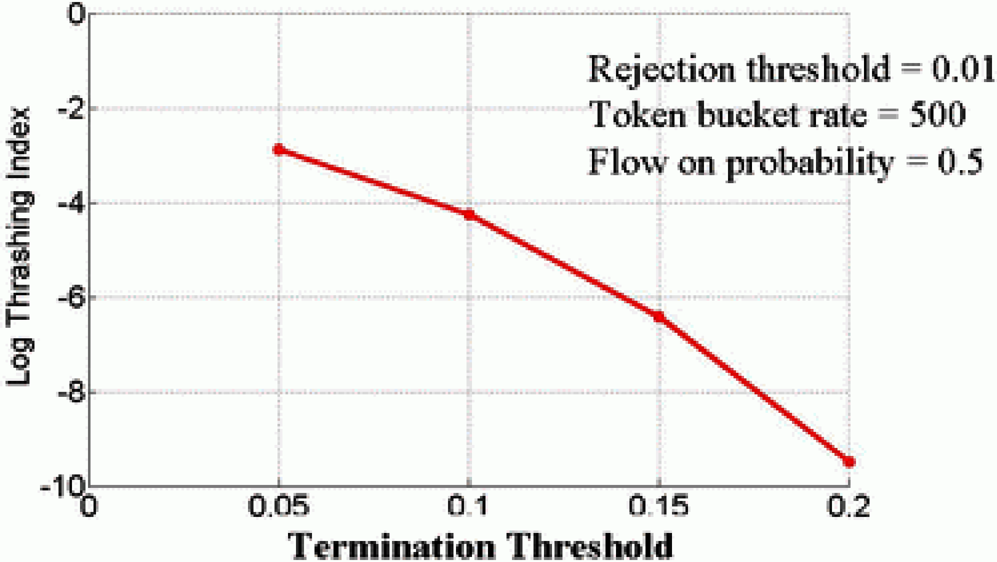
- **Conclusion**: Keeping the termination threshold much higher than the rejection threshold helps avoid threshing

Sensitivity to Term. Threshold (Cont)



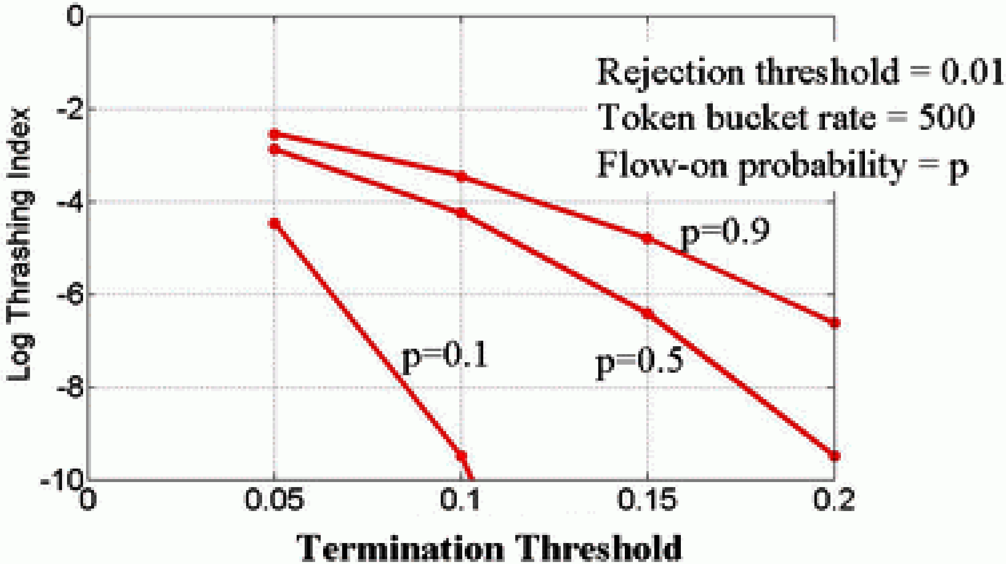
- **Conclusion**: Threshing region decreases as the termination threshold is set farther from the rejection threshold

Sensitivity to Term. Threshold (Cont)



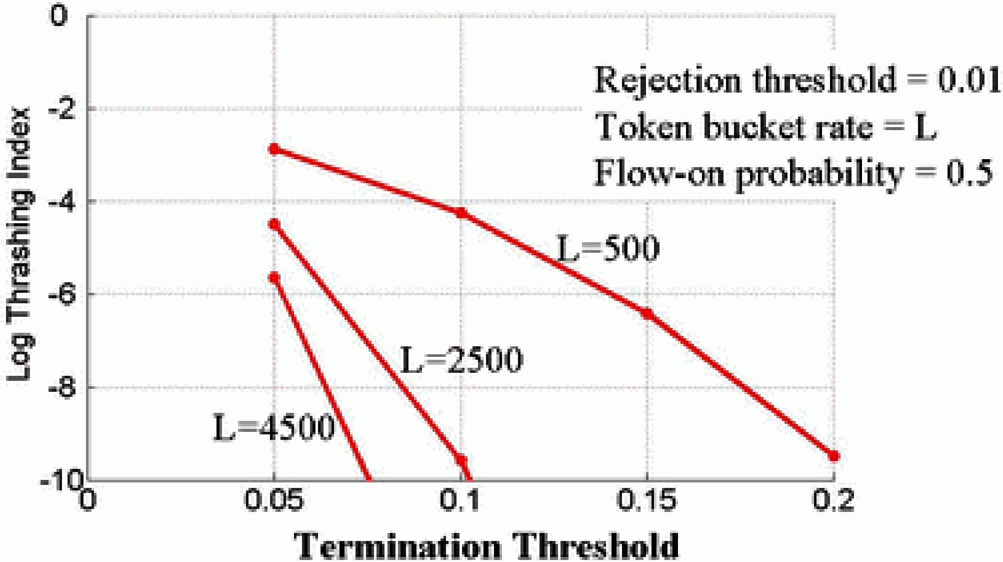
- **Conclusion:** Termination threshold of 0.15 reduces thrashing index to below 10^{-6} for this case.

Sensitivity to Flow-On Probability



- **Conclusion**: If the flows are on more often, the termination threshold has to be set higher

Sensitivity to Token Bucket Rate



- **Conclusion**: For large capacity links, the termination threshold can be set closer to rejection threshold

Summary

1. A closed form expression for flow rejection probability and flow termination probability for single marker case
2. The model explains the thrashing behavior when the system reaches rejection/termination threshold region
3. Thrashing Index = Max{P(Acceptance)×P(Termination)}
4. The termination threshold should be set 10-15% above rejection threshold to avoid thrashing.
5. The difference can be less if the number of flows is larger (large capacity links) or if the flow-on probability is smaller (inactive flows).

Reference

- R. Jain, "The Art of Computer Systems Performance Analysis: Techniques for Experimental Design, Measurement, Simulation, and Modeling," Wiley-Interscience, New York, NY, April 1991, ISBN:0471503361
- Wikipedia, "Error Function," http://en.wikipedia.org/wiki/Error_function